CMB anisotropies from primordial inhomogeneous magnetic fields

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Primordial inhomogeneous magnetic fields of the right strength can leave a signature on the CMB temperature anisotropy and polarization. Potentially observable contributions to polarization B-modes are generated by vorticity and gravitational waves sourced by the magnetic anisotropic stress. We compute the corresponding CMB transfer functions in detail including the effect of neutrinos. The shear rapidly causes the neutrino anisotropic stress to cancel the stress from the magnetic field, suppressing the production of gravitational waves and vorticity on super-horizon scales after neutrino decoupling. A significant large scale signal from tensor modes can only be produced before neutrino decoupling, and the actual amplitude is somewhat uncertain. Plausible values suggest primordial nearly scale invariant fields with $B_\lambda \sim 10^{-10}$G today may be observable from their large scale tensor anisotropy. They can be distinguished from primordial gravitational waves by their non-Gaussianity. Vector mode vorticity sources B-mode power on much smaller scales with a power spectrum somewhat similar to that expected from weak lensing, suggesting amplitudes $B_\lambda \sim 10^{-9}$G may be observable on small scales for a spectral index $n \sim -2.9$. In the appendix we review the covariant equations for computing the vector and tensor CMB power spectra that we implement numerically.

I. INTRODUCTION

Magnetic fields are ubiquitous in the universe, with $\sim 10^{-6}$G coherent fields observed on galactic and cluster scales. However their origin is not well understood (for a review see Ref. [1]). Tiny seed fields $\lesssim 10^{-20}$G may have been amplified by a dynamo mechanism to give the much larger fields we now see, though to what extent this process can work in practice is not yet clear [2]. Alternatively initial fields of strength $\sim 10^{-9}$G can give rise to galactic fields of the observed values without a functioning dynamo mechanism. Such fields have potentially interesting observational signatures on the CMB, and if present would provide powerful constraints on models of the early universe. The absence of such signatures may also serve as a consistency check on models of galaxy evolution that would be observationally incompatible with initial fields this large.

A primordial field of $\sim 10^{-9}$G can leave a signature in the B-mode (curl-like) CMB polarization. Since the scalar (density) perturbations do not produce B-modes at linear order, the B-modes are a much cleaner signal of additional physics than very small fractional changes to the temperature or E-mode polarization. However B-modes are produced at second order through weak lensing [3, 4], and are also generated by primordial gravitational waves (tensor modes) [5]. Other possible sources include topological defects [6, 7]. The focus of many future CMB observations will be on observing the B-modes, so it is useful to assess in detail the various possible components and how they can be distinguished from each other.

Primordial fields with a blue spectrum compatible with nucleosynthesis are far too weak on cosmological scales to leave an interesting signature [8]. In this paper we consider in detail the CMB signal expected from $\sim 10^{-9}$G primordial fields with a nearly scale invariant spectrum. Such fields are not well motivated by current theoretical models, which mostly give much smaller amplitudes or a much bluer spectrum [1, 9, 10, 11]. Observation of primordial fields at this level would therefore be a powerful way to rule out many models. However Ref. [12] present one model in which observably interesting CMB signatures may be produced.

Previous semi-analytical work has investigated the CMB signal from both tensor [13, 14] and vector modes [14, 15] sourced by magnetic fields. Here we give a more detailed numerical analysis of the full linearized equations. We include the effect of neutrinos as they change the way that magnetic fields source gravitational waves and vorticity on super-horizon scales. Our final CMB power spectra include the contribution to the B-mode signal from both the tensor and vector modes. We do not consider helical modes [16, 17] which can be detected via their parity-violating correlations, nor the effects of Faraday rotation [18, 19] (which can be identified by the frequency dependence). We also assume that reionization is relatively sharp and unmodified by energy injection into the IGM from decay of the small scale magnetic field [20]. For a discussion of constraints on homogeneous magnetic fields see e.g. Ref. [21] and references therein.

II. COVARIANT EQUATIONS

We consider linear perturbations in a flat FRW universe evolving according to general relativity with a cosmological constant and cold dark matter, and approximate the neutrinos as massless. Perturbations can be described covariantly in terms of a 3+1 decomposition with respect to some choice of observer velocity u_a (we use natural units, and the signature where $u_a u^a = 1$),

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following Refs. [22, 23, 24, 25]. The stress-energy tensor can be decomposed with respect to u_a as

$$T_{ab} = \rho u_a u_b - p h_{ab} + 2u_{(a}q_{b)} + \pi_{ab} \tag{1}$$

where ρ is the energy density, p is the pressure, q_a is the heat flux and and $\pi_{ab} \equiv T_{\langle ab\rangle}$ is the anisotropic stress. Angle brackets around indices denote the projected (orthogonal to u_a) symmetric trace-tree part (PSTF). The tensor

$$h_{ij} \equiv g_{ij} - u_i u_j, \tag{2}$$

where g_{ij} is the metric tensor, projects into the instantaneous rest space orthogonal to u_a. It defines a spatial derivative $\mathrm{D}_a \equiv h_a{}^b\nabla_b$ orthogonal to u_a where ∇_a is the covariant derivative. Spatial derivatives can be used to quantify perturbations to background quantities, for example the pressure perturbation can be described covariantly in terms of $\mathrm{D}_a p$.

Conservation of total stress-energy $\nabla^a T_{ab} = 0$ implies an evolution equation for the total heat flux q_a

$$\dot{q}_a + \frac{4}{3}\Theta q_a + (\rho+p)A_a - \mathrm{D}_a p + \mathrm{D}^b\pi_{ab} = 0 \tag{3}$$

where $\dot{q}_a \equiv u^b\nabla_b q_a$, $\Theta \equiv \nabla^a u_a$ is three times the Hubble expansion, and $A_a \equiv u_b\nabla^b u_a$ is the acceleration. The evolution equation for the heat flux $q^i_a = (\rho^i + p^i)v^i_a$ of each matter component present is of the form

$$\dot{q}^i_a + \frac{4}{3}\Theta q^i_a + (\rho^i+p^i)A_a - \mathrm{D}_a p^i + \mathrm{D}^b\pi^i_{ab} = L^i_a \tag{4}$$

where L^i_a is an interaction force term. Conservation of total stress energy $q_a = \sum_i q^i_a$ implies that $\sum_i L^i_a = 0$. For magnetic fields the components of the stress-energy tensor are given by[1]

$$\begin{aligned} \rho^B &= 3p^B = -\frac{1}{2}(E^2+B^2) && (5)\\ \pi^B_{ab} &= -E_{\langle a}E_{b\rangle} - B_{\langle a}B_{b\rangle} && (6)\\ q^B_a &= -(E\times B)_a && (7)\end{aligned}$$

where E_a and B_a are the electric and magnetic field projected vectors. We take B^2 and E^2 to be first order, and to this order E_a and B_a are frame invariant. For most of its evolution the universe is a good conductor so we may take $E_a = 0$ in all linear frames: the magnetic fields are frozen in, and in this approximation almost all the complications of MHD disappear. The linearized Bianchi identity for the electromagnetic field tensor implies

$$\dot{B}_a + \frac{2}{3}\Theta B_a = 0 \tag{8}$$

so the magnetic field simply redshifts as $1/S^2$ where S is the scale factor, and hence $\pi^B_{ab} \propto \rho^B \propto 1/S^4$. More general equations can be found in e.g. Ref. [25].

The Poynting vector heat flux is zero ($q^B_a = 0$) in all linear frames because we have set $E_a = 0$. Since A_a is first order, on linearizing we have the Lorentz force L^B_a given by the evolution equation (4)

$$\mathrm{D}^b\pi^B_{ab} - \mathrm{D}_a p^B = L^B_a. \tag{9}$$

This is consistent with the usual $\mathrm{curl}\, B \times B$ expression. The opposite force acts on the baryons to ensure total momentum conservation, which with the Thomson scattering terms [24] gives the baryon velocity evolution equation:

$$\begin{aligned} &\dot{v}_a + \frac{1}{3}\Theta v_a + A_a - \frac{D_a p^b}{\rho^b} = \\ &\quad -\frac{\rho^\gamma}{\rho^b}\left[n_e\sigma_T\left(\frac{4}{3}v_a - I_a\right) + \frac{\mathrm{D}^b\pi^B_{ab} - \mathrm{D}_a p^B}{\rho^\gamma}\right] \end{aligned} \tag{10}$$

where n_e is the electron number density, σ_T is the Thomson scattering cross-section, ρ^γ is the photon energy density, and we neglect baryon pressure terms of the form $p^{b'}/\rho^{b'} \ll 1$. Thus magnetic fields source baryon vorticity, as well as providing extra density and pressure perturbations, and anisotropic stresses.

We define the vorticity vector $\Omega_a \equiv \mathrm{curl}\, u_a$ where for a general tensor

$$\mathrm{curl}\, X_{a_1\ldots a_l} \equiv \eta_{bcd(a_1}u^b\mathrm{D}^c X^d{}_{a_2\ldots a_l)} \tag{11}$$

and round brackets denote symmetrization. It is transverse $\mathrm{D}^a\Omega_a = 0$. Remaining quantities we shall need are[2] the 'electric' E_{ab} and 'magnetic' H_{ab} parts of the Weyl tensor C_{abcd}

$$E_{ab} \equiv C_{acbd}u^c u^d \qquad H_{ab} \equiv \frac{1}{2}\eta_{acdf}C_{be}{}^{cd}u^e u^f \tag{12}$$

(which are frame invariant) and the shear $\sigma_{ab} \equiv \mathrm{D}_{\langle a}u_{b\rangle}$. The Einstein equation and the Bianchi identity give the constraint equations

$$\begin{aligned} \mathrm{D}^a\sigma_{ab} - \frac{1}{2}\mathrm{curl}\,\Omega_b - \frac{2}{3}\mathrm{D}_b\Theta - \kappa q_b &= 0\\ \mathrm{D}^a E_{ab} - \kappa\left(\frac{\Theta}{3}q_b + \frac{1}{3}\mathrm{D}_b\rho + \frac{1}{2}\mathrm{D}^a\pi_{ab}\right) &= 0\\ \mathrm{D}^a H_{ab} - \frac{1}{2}\kappa[(\rho+p)\Omega_b + \mathrm{curl}\, q_b] &= 0\\ H_{ab} - \mathrm{curl}\,\sigma_{ab} + \frac{1}{2}\mathrm{D}_{\langle a}\Omega_{b\rangle} &= 0, \end{aligned} \tag{13}$$

[1] Note that unlike many other authors we use natural rather than Gaussian units. Due to the signature choice $-E^2 \geq 0$.

[2] From here on we do not use E for the electric field; E_{ab} has nothing to do with electromagnetism, and is merely called the analogous 'electric' part of the Weyl tensor by analogy. The 'electric' part of the polarization distribution is written as $\mathcal{E}_{A_l}$.

and the evolution equations

$$\begin{aligned}
\dot{\Omega}_a + \frac{2}{3}\Theta\Omega_a &= \operatorname{curl} A_a \\
\dot{\sigma}_{ab} + \frac{2}{3}\Theta\sigma_{ab} &= -E_{ab} - \frac{1}{2}\kappa\pi_{ab} \\
\dot{E}_{ab} + \Theta E_{ab} &= \operatorname{curl} H_{ab} + \frac{\kappa}{2}\left[\dot{\pi}_{ab} - (\rho+p)\sigma_{ab} + \frac{\Theta}{3}\pi_{ab}\right] \\
\dot{H}_{ab} + \Theta H_{ab} &= -\operatorname{curl} E_{ab} - \frac{\kappa}{2}\operatorname{curl}\pi_{ab}.
\end{aligned} \tag{14}$$

Here $\kappa \equiv 8\pi G$.

The distribution functions for the various species can be expanded into multipole moments. For example the photon multipole tensors $I_{A_l} \equiv I_{\langle a_1 \ldots a_l \rangle}$ are defined as moments of the distribution of the photon intensity $I(e)$ per solid angle as [26]

$$I_{A_l} \equiv \int d\Omega_e \, I(e) \; e_{\langle A_l \rangle}, \tag{15}$$

where the direction vector e_a is normalized to $e^a e_a = -1$ and $e_{\langle A_l \rangle} = e_{\langle a_1} \ldots e_{a_l \rangle}$ are irreducible PSTF tensors. The $e_{\langle A_l \rangle}$ are orthogonal:

$$\frac{1}{4\pi}\int d\Omega_e \, e_{\langle A_l \rangle} e^{\langle B_n \rangle} = \delta_{ln} \frac{(-2)^l (l!)^2}{(2l+1)!} h_{\langle a_1}^{\langle b_1} \ldots h_{a_l \rangle}^{b_l \rangle}. \tag{16}$$

The I_{A_l} multipole tensors have $2l+1$ degrees of freedom, $I_{ab} = \pi^\gamma_{ab}$ is the anisotropic stress, $I_a = q^\gamma_a$ is the heat flux and $I = \rho_\gamma$. The temperature anisotropy can then be expanded as

$$\frac{\Delta T(e)}{T} = \sum_l \frac{(2l+1)!}{4(-2)^l (l!)^2} \frac{I_{A_l} e^{A_l}}{\rho_\gamma} = \sum_l \sum_{m=-l}^{l} a_{lm} Y_{lm}(e) \tag{17}$$

where the latter expansion in terms of spherical harmonics Y_{lm} is the non-covariant version of the expansion in I_{A_l}. The CMB power spectrum is defined in terms of the variance of the spherical harmonic components a_{lm} by

$$C_l \equiv \langle |a_{lm}|^2 \rangle = \frac{\pi}{4} \frac{(2l)!}{(-2)^l (l!)^2} \frac{\langle I_{A_l} I^{A_l} \rangle}{\rho_\gamma^2}. \tag{18}$$

Analogous results for the polarization are given in Ref. [27], where $\mathcal{E}_{A_l}$ is a gradient-like multipole of the polarization tensor and $\mathcal{B}_{A_l}$ is a curl-like multipole.

The covariant equations can be expanded in terms of scalar, vector and tensor harmonics. The details and definitions are given in the appendix. In the following sections we analyse in detail the tensor and vector equations, where each quantity is a component of a harmonic expansion, and k labels are suppressed. Scalar modes can source temperature and E-polarization CMB signals, however since we are mostly interested in the B-polarization signal, which is not sourced by scalar modes, we do not discuss scalar modes here. A partial analysis of scalar modes is given in Ref. [28].

III. TENSORS

Expanding in $m=2$ tensor harmonics (A12) (and suppressing m indices), the constraint equations (13) imply that the Weyl tensor variable H is related to the shear by $H = \sigma$. The evolution equations (14) then give

$$\begin{aligned}
k^2(E' + \mathcal{H}E) - k^3\sigma + \frac{\kappa}{2}S^2(\rho+p)k\sigma &= \frac{\kappa}{2}S^2(\Pi' + \mathcal{H}\Pi) \\
\sigma' + 2\mathcal{H}\sigma + kE &= -\frac{\kappa S^2 \Pi}{2k}
\end{aligned} \tag{19}$$

where primes denote derivatives with respect to conformal time η and $\mathcal{H} \equiv S\Theta/3$ is the conformal Hubble parameter. The Weyl tensor variable E and the shear σ define the new variable

$$H_T \equiv -2E - \frac{\sigma'}{k} \tag{20}$$

to correspond to the metric perturbation variable of non-covariant approaches. It satisfies $H_T' = -k\sigma$, and the above equations combine to give the well known evolution equation

$$H_T'' + 2\mathcal{H}H_T' + k^2 H_T = \kappa S^2 \Pi. \tag{21}$$

Magnetic fields provide a component of the anisotropic stress Π and hence source gravitational waves, and we quantify the magnetic field source by the dimensionless ratio $B_0 \equiv \Pi_B/\rho_\gamma$. The covariant tensor equations are discussed in more detail in Ref. [26].

Equations for the evolution of the tensor multipoles are obtained from the appendix ($m=2$ in Eqs. (A13) and (A14)). We use a series expansion in conformal time η to identify the regular primordial modes in the early radiation dominated era. Defining $\omega \equiv \Omega_m \mathcal{H}_0/\sqrt{\Omega_R}$, where $\Omega_R = \Omega_\gamma + \Omega_\nu$, and $\mathcal{H}_0$ and Ω_i are the Hubble parameter and densities (in units of the critical density) today, the Friedmann equation gives

$$S = \frac{\Omega_m \mathcal{H}_0^2}{\omega^2}\left(\omega\eta + \frac{1}{4}\omega^2\eta^2 + \mathcal{O}(\eta^5)\right). \tag{22}$$

Defining the ratios $R_\nu \equiv \Omega_\nu/\Omega_R$, $R_\gamma \equiv \Omega_\gamma/\Omega_R$ and keeping lowest order terms the regular solution (with zero initial anisotropies for $l > 2$) is

$$\begin{aligned}
H_T &= H_T^{(0)}\left(1 - \frac{5}{2}\frac{(k\eta)^2}{4R_\nu + 15}\right) + \frac{15}{28}\frac{R_\gamma B_0 (k\eta)^2}{4R_\nu + 15} \\
\sigma &= \frac{5H_T^{(0)} k\eta}{4R_\nu + 15} - \frac{15}{14}\frac{R_\gamma B_0 k\eta}{4R_\nu + 15} \\
\pi_\nu &= -\frac{R_\gamma B_0}{R_\nu}\left(1 - \frac{15}{14}\frac{(k\eta)^2}{4R_\nu + 15}\right) + \frac{4}{3}\frac{(k\eta)^2 H_T^{(0)}}{4R_\nu + 15}
\end{aligned} \tag{23}$$

where $H_T^{(0)}$ is the initial value (after neutrino decoupling) and $\pi_\nu \equiv \Pi_\nu/\rho_\nu$. The $B_0 \neq 0$ mode (with $H_T^{(0)} = 0$) has compensating anisotropic stresses: the sum of the

magnetic and neutrino terms gives the total source term

$$\kappa S^2\Pi = \frac{45}{14}\frac{R_\gamma k^2 B_0}{4R_\nu+15}\left(1-\frac{(45-2R_\nu)\omega\eta}{2R_\nu+15}\right)+\mathcal{O}(\eta^2) \tag{24}$$

rather than the $\propto S^2\rho_\gamma \propto 1/\eta^2$ result expected without collisionless radiation. For $k \ll \mathcal{H}$ there is therefore no sourcing of gravitational waves during radiation domination, so $H_T \propto (k\eta)^2$ if it was zero initially. Collisionless fluids suppress generation of gravitational waves on super-horizon scales.

Before neutrino decoupling there is no neutrino anisotropic stress so the magnetic field source is not compensated. Taking η_{in} as the magnetic field production time (at which we take $H_T = \sigma = 0$), the solution for $k\eta \ll 1$ is approximately[3]

$$H_T \approx 3R_\gamma B_0\left(\ln(\eta/\eta_{\rm in})+\frac{\eta_{in}}{\eta}-1\right) \tag{25}$$

$$\sigma \approx -\frac{3R_\gamma B_0}{k\eta}\left(1-\frac{\eta_{in}}{\eta}\right). \tag{26}$$

After neutrino decoupling this mode must convert into a combination of the above regular modes, which can then be used to compute the observable signature. As the neutrino coupling is switched off the neutrino anisotropic stress becomes important, and with no scattering evolves as

$$\pi_\nu' = -\frac{k}{3}J_3 + \frac{8}{15}k\sigma. \tag{27}$$

Since the octopole J_3 and π_ν will be zero before neutrino decoupling, for modes well outside the horizon we have $\pi_\nu' \sim -B_0/\eta$ just after decoupling (we assume $\eta \gg \eta_{in}$), and hence the neutrino anisotropic stress grows logarithmically $\pi_\nu \sim -B_0\ln(\eta/\eta_*^\nu)$. It therefore reaches the constant value $\pi_\nu \sim -B_0$ of the regular solution in about one e-folding. At this point H_T ceases to grow logarithmically because the magnetic anisotropic stress source is now cancelled by the neutrinos, and H_T gives the amplitude of the usual regular solution $H_T^{(0)}$.

Thus after neutrino decoupling we expect a combination of the usual passive primordial tensor mode and the regular compensated sourced mode. As we show explicitly below, the compensated mode can be neglected compared to the small scale vector mode contribution. The passive tensor mode has

$$H_T^{(0)} \approx 3R_\gamma B_0 \ln(\eta_*^\nu/\eta_{\rm in}) \tag{28}$$

where η_*^ν is the time of neutrino decoupling (assuming magnetic field generation is during radiation domination at $\eta \sim \eta_{\rm in}$ and before neutrino decoupling). Since the transverse traceless part of the metric tensor $h_{ij} = \sum_{k,\pm} 2H_T Q_{ij}^2$ this corresponds to a primordial power spectrum for h of

$$P_h \approx \left[6R_\gamma \ln(\eta_*^\nu/\eta_{\rm in})\right]^2 P_{B_0}. \tag{29}$$

Power spectra are defined in Eq. (A26) and $R_\gamma \sim 0.6$.

Thus the tensor covariance part of the magnetic field signal is very similar to that expected from primordial gravitational waves, and can be computed trivially by using the above tensor power spectrum in the numerical codes CAMB [29] or CMBFAST [30]. However unlike primordial gravitational waves the spectral index of P_h here is expected to be at least slightly blue, and the signal should be non-Gaussian because B_0 is quadratic in the magnetic field. Allowing for subtracting the B-mode lensing signal, levels of $P_h \sim 10^{-15}$ may be observable[4] [31]. This corresponds to $B_\lambda \sim 10^{-10}$G for $\eta_{\rm in}/\eta_*^\nu \sim 10^{-6}$ and a close to scale invariant spectrum. To distinguish this from primordial gravitational waves from inflation one would need to detect the non-Gaussianity or small scale power from the vector modes.

The compensation mechanism in principle works with any collisionless relativistic fluid, even when it only makes up a fraction $R_\nu \to 0$ of the energy density (R_ν can be interpreted as any collisionless component). However if there is collisionless relativistic matter only from a time η_* after magnetic field production, partial compensation will be effective at a time $\eta \sim \eta_* e^{R_\gamma/R_\nu}$. For small fractions R_ν this is a very large time, so the mechanism is inefficient for components that are only a small fraction of the density allowed by the nucleosynthesis bound. Neutrinos themselves can only suppress gravitational wave production after neutrino decoupling at $z \sim 10^9$.

IV. VECTORS

Unlike tensors modes, vectors are not in general frame invariant. It is therefore convenient to choose the frame u_a to simplify the analysis. At linear order one can always write $u_a = u_a^\perp + v_a$, where $u_a^\perp$ is hypersurface orthogonal and v_a is first order, so $\operatorname{curl} u_a = \operatorname{curl} v_a$. For a zero order scalar quantity X it follows that

$$\mathrm{D}_a X = \mathrm{D}_a^\perp X - v_a \dot{X}. \tag{30}$$

For vector modes $(\mathrm{D}_a^\perp X)^{(1)} = 0$, and it is convenient to choose the frame u_a to be hypersurface orthogonal so that $\operatorname{curl} u_a = 0$ and hence $(\bar{\mathrm{D}}_a X)^{(1)} = 0$, where the bar denotes evaluation in the zero vorticity frame. From its

[3] It appears that σ will be large when $k\eta \ll 1$, however the contribution to $\langle\sigma_{ab}\sigma^{ab}\rangle/(\kappa\rho_\gamma)$ and similar terms remain small, so this does not violate the linearity assumption.

[4] Observable in the sense that in the null hypothesis that there are no non-lensing B-modes, the residual noise level would be consistent with this level. Actual subtraction of CMB lensing in the presence of B-modes from magnetic field sourced vector modes may be extremely difficult, but one should still be able to detect violations of the null hypothesis that there are none.

propagation equation, vanishing of the vorticity also implies that $\bar{A}_a^{(1)}=0$, so the zero vorticity frame coincides with the synchronous gauge. The CDM velocity is also zero in this frame modulo a mode that decays as $1/S$ where S is the scale factor.

Expanding in the $m=1$ vector harmonics (A12) the equations for the harmonic coefficients in the zero vorticity frame ($\Omega=0$) reduce to

$$k(\bar{\sigma}'+2\mathcal{H}\bar{\sigma})=-\kappa S^2\Pi$$
$$H=\frac{1}{2}\bar{\sigma} \qquad 2\kappa S^2\bar{q}=k^2\bar{\sigma}. \tag{31}$$

The combination $v+\sigma$ (the Newtonian gauge velocity) is frame invariant, as are $\bar{\sigma}=\sigma+\Omega$ and $\bar{v}=v-\Omega$. By choosing to consider the zero vorticity frame we have simply expedited the derivation of the above frame invariant equations. Other papers use the Newtonian gauge [32], in which $\bar{\sigma}$ is the vorticity. The equations are consistent. The baryon vorticity evolution equation (10) becomes

$$\bar{v}'+\mathcal{H}\bar{v}=-\frac{\rho_\gamma}{\rho_b}\left[Sn_e\sigma_T\left(\frac{4}{3}v-I_1\right)+\frac{1}{2}kB_0\right] \tag{32}$$

where $I_1=4v_\gamma/3$ and as before $B_0\equiv\Pi_B/\rho_\gamma$.

At early times the baryons and photons are tightly coupled, the opacity $\tau_c^{-1}\equiv S\mathrm{n}_e\sigma_\mathrm{T}$ is large. This means $v_\gamma\approx v$, and we can do an expansion in τ_c that is valid for $\epsilon\equiv\max(k\tau_c,\mathcal{H}\tau_c)\ll 1$. To lowest order the baryon velocity evolves as

$$\bar{v}' \;=\; -\frac{R}{1+R}\left(\mathcal{H}\bar{v}+\frac{3}{8}\frac{kB_0}{R}\right)+\mathcal{O}(\tau_c) \tag{33}$$

where $R\equiv 3\rho_b/4\rho_\gamma$. The solution with zero initial vorticity is

$$\bar{v}\approx-\frac{3}{8}\frac{B_0k\eta}{1+R}, \tag{34}$$

so the magnetic field sources a growing baryon vorticity. At matter domination the vorticity starts to redshift away, however by recombination this will only be an order unity effect. On smaller scales where $k\tau_c=\mathcal{O}(1)$ before decoupling the perturbations are damped by photon diffusion, giving a characteristic fall off in perturbation power on small scales.

To identify the primordial regular mode we perform a series expansion as we did for the tensor case. Assuming no primordial radiation vorticity (the regular vector mode [33]) the result is

$$\bar{\sigma} \;=\; -\frac{45}{14}k\eta\frac{R_\gamma B_0}{4R_\nu+15} \tag{35}$$
$$\bar{q}_\gamma \;=\; -\frac{k\eta B_0}{2} \tag{36}$$
$$\bar{q}_\nu \;=\; \frac{k\eta B_0}{2}\frac{R_\gamma}{R_\nu} \tag{37}$$
$$\pi_\nu \;=\; -\frac{R_\gamma}{R_\nu}B_0. \tag{38}$$

As in the tensor case the anisotropic stresses compensate on super-horizon scales, so they source negligible shear $\bar{\sigma}$ on these scales. However, unlike the tensor case, any non-zero $\bar{\sigma}$ present on super-horizon scales at neutrino decoupling decays away rapidly and has no observable effect, so the evolution prior to neutrino decoupling is irrelevant. The observable signature comes from the vorticity sourced on sub-horizon scales by the magnetic anisiotropic stress on its own.

In the approximation that recombination is sharp at $\eta=\eta^*$, the photon multipoles are given approximately from the integral solution (A16) by

$$\frac{I_l(\eta_0)}{4}\approx\left[(\bar{v}+\bar{\sigma})\Psi_l+\frac{\zeta}{4}\frac{d\Psi_l}{d\chi}\right]_{\eta*}+\int_{\eta*}^{\eta_0}d\eta\bar{\sigma}'\Psi_l, \tag{39}$$

where $\Psi_l\equiv lj_l(\chi)/\chi$, $j_l(x)$ is a spherical Bessel function and $\chi\equiv k(\eta_0-\eta)$. Thus the Newtonian gauge vorticity $\bar{\sigma}$ has last scattering and integrated Saches-Wolfe (ISW) contributions to the temperature anisotropy. In the absence of neutrino anisotropic stress $\bar{\sigma}\sim-3R_\gamma B_0/(k\eta)$ during radiation domination, as in the tensor case. On super-horizon scales this is large, and would give a large scale contribution from $\bar{\sigma}$ orders of magnitude larger than the small super-horizon Doppler contribution from $\bar{v}$. Previous work [14, 34] has neglected the contributions from $\bar{\sigma}$ around last scattering, an approximation that is good on small scales. However on super-horizon scales when $\bar{\sigma}$ is large, the diverging $\bar{\sigma}$ contributions are far from negligible. Previous power spectra are approximately the right shape on large scales, but for the wrong reason: the large scale anisotropies are small because of neutrino compensation suppressing the source term for $\bar{\sigma}$, not because they are insensitive to $\bar{\sigma}$. Similar comments apply to the polarization power spectra.

There is one caveat to the above. The decay of $\bar{\sigma}$ from neutrino decoupling to last scattering amounts to a decay factor of about $(z_*^\nu/z_*^\gamma)^2\sim 10^{12}$. However on arbitrarily large scales the $1/(k\eta_*^\nu)$ evolution of $\bar{\sigma}$ before neutrino decoupling can be larger than this, so on the very largest scales there can still be a contribution to $\bar{\sigma}$ at last scattering. In the asymptotic limit the dipole I_1 appears singular, though the quadrupole I_2 is finite. The constraint $n>-3$ ensures that the total power from large scale modes is not singular if the individual modes are not. Here we neglect this effect, effectively assuming the power spectrum cuts off on sufficiently large scales that are otherwise unobservable. It is unclear whether there is a serious infrared problem with a nearly scale invariant spectrum or not. We simply compute the CMB transfer function from a given anisotropy stress B_0, and defer the issue of what the spectrum actually is, how it could be generated, and its actual asymptotic behaviour.

V. NUMERICAL RESULTS

In this paper the focus is on calculation of accurate CMB transfer functions from a given initial distribution.

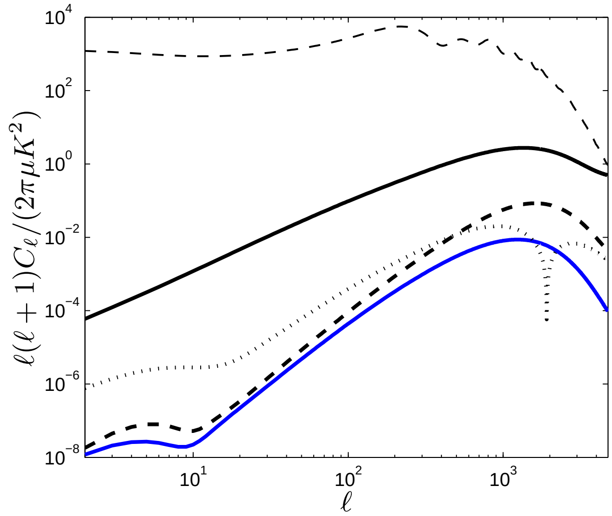

FIG. 1: Typical CMB temperature TT (top solid), polarization EE (bottom solid), BB (dashed thick) and cross-correlation TE (dotted; absolute value) power spectra from vector modes with $B_\lambda = 3\times10^{-9}$G, $n=-2.9$. The thin dashed line shows the scalar adiabatic mode TT spectrum (without magnetic fields). The increase in power at $\ell \lesssim 10$ is due to reionization at redshift $z\sim13$.

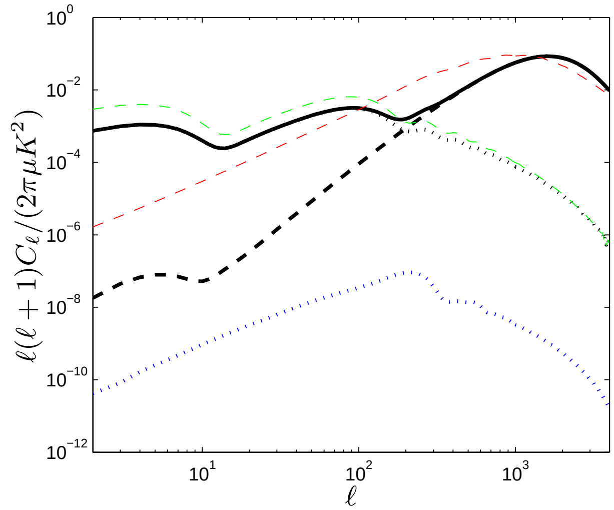

FIG. 2: Typical CMB B-mode polarization power spectra spectra for vector modes (thick dashed), tensor modes (dotted) and total (solid) for $B_\lambda = 3\times10^{-9}$G, $n=-2.9$, $\eta_*^\nu/\eta_{\rm in} = 10^6$. The bottom dotted line is from the negligible tensor modes sourced after neutrino decoupling (the compensated mode). The top dotted line is from tensors sourced after magnetic field generation until neutrino decoupling (and should be regarded as an estimate correct to a few orders of magnitude). The thin dashed lines show the B-mode spectrum from weak lensing (peaking at $\ell\sim1000$), and scale invariant primordial tensors with initial power ratio $\sim10^{-1}$ (peaking at $\ell\sim100$). The magnetic field spectra scale as B_λ^4.

As a convenient ansatz for computing sample C_l power spectra we assume a Gaussian primordial magnetic field distribution, with power spectrum $P_B \propto k^{n-3}$ (the definition of n is conventional).

One can define a smoothed magnetic field B_λ using a Gaussian smoothing of width λ (we choose $\lambda=1$Mpc) and express the power spectrum in terms of B_λ as in Refs. [13, 14]. In harmonic space the anisotropic stress is given by a convolution of the underlying magnetic fields, so the power spectrum for the anisotropic stress at a given k feels the power from across the P_B spectrum. For vectors and tensors the resultant power spectrum is given approximately by [14]

$$P_{B_0} \approx \frac{4}{(2n+3)}\left[\frac{(2\pi)^{n+3}B_\lambda^2}{2\Gamma(\frac{n+3}{2})\rho_\gamma}\right]^2 \times \left\{\left(\frac{k_D}{k_\lambda}\right)^{2n+3}\left(\frac{k}{k_\lambda}\right)^3 + \frac{n}{n+3}\left(\frac{k}{k_\lambda}\right)^{2n+6}\right\} \quad (40)$$

for $-3<n$. The scale k_D comes from a small scale damping cut-off [13, 14], and does not affect the power spectrum significantly for nearly scale invariant power spectra with $n\sim-3$. The spectrum is singular at $n=-3$, which comes from the singular build up of super-horizon power for a scale invariant B spectrum with no large scale cut-off. Since $B_0 \equiv \Pi/\rho_\gamma$ is quadratic in B, the spectrum of B_0 will be non-Gaussian, so the power spectrum only contains a subset of the available information, though it is useful to assess the detectability amplitude.

As a sample example we take $B_\lambda = 3\times10^{-9}$G and $n=-2.9$ (as in Ref. [15]), which implies

$$P_{B_0} \approx 1.16\times10^{-13}\left(\frac{k}{k_\lambda}\right)^{0.2}. \quad (41)$$

Since data will only constrain P_{B_0} directly, we take this equation to be exact for our numerical results so they may easily be related to different power spectra for P_{B_0} (which may or may not come from the assumed power law spectrum of B_a fluctuations). The power spectrum P_{B_0} scales as B_λ^4 as do the CMB power spectra, so large changes in overall amplitude can be obtained from relatively small changes in the primordial field: the value of B_λ has to be quite finely tuned to give a CMB signature that is neither totally dominant nor totally negligible.

Numerical results from vector modes with $B_\lambda = 3\times10^{-9}$G are shown in Fig. 1, in comparison with the spectra expected from primordial curvature perturbations and possible primordial gravitational waves. For this B_λ the effect on the temperature power spectrum is negligible; only if $B_\lambda \gtrsim 8\times10^{-9}$G could there be a significant contribution to the power at $l\gtrsim2000$, perhaps contributing some of the power observed on these scales [35].

The contributions to the most easily distinguished B-modes are shown in Fig. 2, including the tensor contribution. It is clear that the compensated tensor mode has a

negligible observational signature and can be neglected. The exact amplitude of the large scale tensor signal from gravitational waves sourced before neutrino decoupling is uncertain because we do not know the time (or mechanism) of field generation, nor have we modelled neutrino decoupling in detail.

Our vector mode results are in broad agreement with the semi-analytical results of [15]. The main qualitative difference is that our TE cross-correlation changes sign in the damping region. The quantitative results differ somewhat across the spectrum due to our more detailed full analysis of the damping, recombination, inclusion of neutrinos and modelling of reionization (we have also used a slightly different primordial power spectrum). The results in Ref. [14] for the large scale vector and tensor spectra are too large by a factor[5] of $(2\pi)^3$ (giving constraints on B_λ too small by a factor of $(2\pi)^{3/4} \sim 4$). There was another normalization error in [36] but corrected in [15]. Refs. [13, 14] provide analytical solutions valid for tensor modes that are super-horizon at decoupling, which give C_l spectra qualitatively valid for $\ell \lesssim 100$. However this approximation was also used for $l < 500$ in Ref. [14], and so their result is qualitatively incorrect at $l \gtrsim 100$. Their tensor polarization results also suffer qualitative problems because the peak in the visibility was neglected. All previous analyses of sourced tensor modes have neglected neutrino compensation, giving results somewhat larger (but the result is still, in any case, somewhat uncertain).

VI. CONCLUSIONS

Nearly scale invariant primordial magnetic fields can give a potentially observable CMB signal if they happen to be $\gtrsim 10^{-10}$G. The observational B-mode signature comes from tensor modes, giving a non-Gaussian spectrum otherwise essentially identical to that expected from primordial gravitational waves, and vector modes giving power on small scales.

Any possible future detection of primordial gravitational waves from large scale B-modes should be carefully distinguished from that produced by magnetic fields. Any primordial signal is expected to be Gaussian, so Gaussianity tests can be used to distinguish them (methods for robustly isolating the B-mode component on sections of the sky are given in Ref. [37], and can be used to construct a set of cut-sky modes that should be Gaussian if they are due to inflation). Low frequency observations may also be able to detect Faraday rotation [19], which would be a clear signal of magnetic fields. Small scale B-mode observations from magnetic fields will need to be carefully distinguished from the weak lensing signal. Regular primordial vector modes, though theoretically unmotivated, can also in principle give a significant small and large scale B-mode signal [33]. They may be distinguished by their sharper fall in power on very small scales due to the lack of sources. Topological defects [6] can be identified by the lack of non-Gaussian tensor mode power on large scales. Note that throughout we have been assuming idealized observations; in practice foregrounds and systematics may well pose very serious problems (see e.g. Ref. [38]).

Our analysis is significantly more detailed than previous work, in that we have numerically solved the full set of linearized equations. There is a qualitatively important mechanism of a neutrino anisotropic stress compensation on super-horizon scales that was previously neglected. Computing the full transfer functions is rather straightforward, and we encourage future workers in this area to at least compare their semi-analytic results with the numerical answer to ensure that important physical effects have not be accidentally overlooked. Our modified version of CAMB[6] [29] for efficiently computing vector mode power spectra is publicly available, and may also be useful for computing anisotropies from other sources, for example topological defects or second order effects.

Acknowledgments

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[5] An inconsistency between their definition Eq. 2.17 and the equation for the C_l, Eq. 5.1

[6] `http://camb.info`

APPENDIX A: MULTIPOLE EQUATIONS, HARMONIC EXPANSION AND C_l

In this appendix we review in a streamlined fashion the multipole equations, solutions, and equations for the C_l needed for numerical calculation [24, 26, 27]. The definitions used here are precisely those used in the CAMB [29] numerical implementation. Equivalent results using the total angular momentum method are given in Ref. [32].

The photon multipole evolution is governed by the geodesic equation and Thomson scattering, giving [27]

$$\dot{I}_{A_l} + \frac{4}{3}\Theta I_{A_l} + D^b I_{bA_l} - \frac{l}{2l+1} D_{\langle a} I_{A_{l-1}\rangle} + \frac{4}{3} I A_{a_1}\delta_{l1} - \frac{8}{15} I \sigma_{a_1 a_2}\delta_{l2} = -n_e\sigma_T\left(I_{A_l} - I\delta_{l0} - \frac{4}{3} I v_{a_1}\delta_{l1} - \frac{2}{15}\zeta_{a_1a_2}\delta_{l2}\right) \quad \text{(A1)}$$

where I_{A_l} is taken to be zero for $l<0$ and

$$\zeta_{ab} \equiv \frac{3}{4} I_{ab} + \frac{9}{2}\mathcal{E}_{ab} \quad \text{(A2)}$$

is a source from the anisotropic stress and E-polarization. The equation for the density perturbation $D_a I$ is obtained by taking the spatial derivative of the above equation for $l=0$. The corresponding evolution equations for the polarization multipole tensors are [27]

$$\begin{aligned}
\dot{\mathcal{E}}_{A_l} + \frac{4}{3}\Theta\mathcal{E}_{A_l} + \frac{(l+3)(l-1)}{(l+1)^2}\mathrm{D}^b\mathcal{E}_{bA_l} - \frac{l}{2l+1}D_{\langle a_l}\mathcal{E}_{A_{l-1}\rangle} - \frac{2}{l+1}\,\mathrm{curl}\,\mathcal{B}_{A_l} &= -n_e\sigma_T(\mathcal{E}_{A_l} - \frac{2}{15}\zeta_{a_1a_2}\delta_{l2}) \\
\dot{\mathcal{B}}_{A_l} + \frac{4}{3}\Theta\mathcal{B}_{A_l} + \frac{(l+3)(l-1)}{(l+1)^2}\mathrm{D}^b\mathcal{B}_{bA_l} - \frac{l}{2l+1}D_{\langle a_l}\mathcal{B}_{A_{l-1}\rangle} + \frac{2}{l+1}\,\mathrm{curl}\,\mathcal{E}_{A_l} &= 0. \quad \text{(A3)}
\end{aligned}$$

For numerical solution we expand the covariant equations into scalar, vector and tensor harmonics. The resulting equations for the modes at each wavenumber can be studied easily and also integrated numerically.

1. Scalar, vector, tensor decomposition

It is useful to do a decomposition into m-type tensors, scalar ($m=0$), vector ($m=1$) and 2-tensor ($m=2$) modes. They describe respectively density perturbations, vorticity and gravitational waves. In general a rank$-\ell$ PSTF tensor X_{A_l} can be written as a sum of $m-$type tensors

$$X_{A_l} = \sum_{m=0}^{l} X_{A_l}^{(m)}. \quad \text{(A4)}$$

Each $X_{A_l}^{(m)}$ can be written in terms of $l-m$ derivatives of a transverse tensor

$$X_{A_l}^{(m)} = D_{\langle A_{l-m}}\Sigma_{A_m\rangle} \quad \text{(A5)}$$

where $\mathrm{D}_{A_l} \equiv \mathrm{D}_{a_1}\mathrm{D}_{a_2}\ldots\mathrm{D}_{a_l}$ and Σ_{A_m} is first order, PSTF and transverse $\mathrm{D}^{a_m}\Sigma^m_{A_{m-1}a_m} = 0$. The 'scalar' component is $X^{(0)}$, the 'vector' component is $X_a^{(1)}$, etc. Since General Relativity gives no sources for X_{A_m} with $m>2$ usually only scalars, vectors and (2-)tensors are considered. At linear order they evolve independently.

2. Harmonic expansion

For numerical work we perform a harmonic expansion in terms of zero order eigenfunctions of the Laplacian $Q^m_{A_m}$,

$$\mathrm{D}^2 Q^m_{A_m} = \frac{k^2}{S^2} Q^m_{A_m}, \quad \text{(A6)}$$

where $Q^m_{A_m}$ is transverse on all its indices, $\mathrm{D}^{a_m}Q^m_{A_{m-1}a_m} = 0$. So a scalar would be expanded in terms of Q^0, vectors in terms of Q^1_a, etc. We usually suppress the labelling of the different harmonics with the same eigenvalue, but when a function depends only on the eigenvalue we write the argument explicitly, e.g. $f(k)$.

For $m>0$ there are eigenfunctions with positive and negative parity, which we can write explicitly as $Q^{m\pm}_{A_m}$ when required. Since

$$\mathrm{D}^2(\,\mathrm{curl}\,Q_{A_m}) = \,\mathrm{curl}\,(\mathrm{D}^2 Q_{A_m}) = \frac{k^2}{S^2}\,\mathrm{curl}\,Q_{A_m} \quad \text{(A7)}$$

they are related by the curl operation. Using the result

$$\mathrm{curl}\;\mathrm{curl}\,Q^m_{A_m} = \frac{k^2}{S^2} Q^m_{A_m} \quad \text{(A8)}$$

we can choose to normalize the $\pm$ harmonics the same way so that

$$\mathrm{curl}\,Q^{m\pm}_{A_m} = \frac{k}{S} Q^{m\mp}_{A_m}. \quad \text{(A9)}$$

A rank-ℓ PSTF tensor of either parity may be constructed from $Q^{m\pm}_{A_m}$ as

$$Q^m_{A_l} \equiv \left(\frac{S}{k}\right)^{l-m} \mathrm{D}_{\langle A_{l-m}} Q^m_{A_m\rangle} \tag{A10}$$

and an $X^{(m)}_{A_l}$ component of X_{A_l} may be expanded in terms of these tensors. They satisfy

$$\begin{aligned} \mathrm{D}^2 Q^m_{A_l} &= \frac{k^2}{S^2} Q^m_{A_l} \\ \mathrm{D}^{a_l} Q^m_{A_{l-1}a_l} &= \frac{k}{S}\frac{(l^2-m^2)}{l(2l-1)} Q^m_{A_{l-1}} \\ \mathrm{curl}\, Q^{m\pm}_{A_l} &= \frac{m}{l}\frac{k}{S} Q^{m\mp}_{A_l} \end{aligned} \tag{A11}$$

where $l \geq m$.

Dimensionless harmonic coefficients are defined by

$$\begin{aligned} \sigma^{(m)}_{ab} &= \sum_{k,\pm} \frac{k}{S}\sigma^{(m)\pm} Q^{m\pm}_{ab} & H^{(m)}_{ab} &= \sum_{k,\pm} \frac{k^2}{S^2} H^{(m)\pm} Q^{m\pm}_{ab} \\ q^{(m)}_a &= \sum_{k,\pm} q^{(m)\pm} Q^{m\pm}_a & E^{(m)}_{ab} &= \sum_{k,\pm} \frac{k^2}{S^2} E^{(m)\pm} Q^{m\pm}_{ab} \\ \pi^{(m)}_{ab} &= \sum_{k,\pm} \Pi^{(m)\pm} Q^{m\pm}_{ab} & I^{(m)}_{A_l} &= \rho_\gamma \sum_{k,\pm} I^{(m)\pm}_l Q^{m\pm}_{A_l} \\ \Omega_a &= \sum_{k,\pm} \frac{k}{S}\Omega^{\pm} Q^{1\pm}_a & A^{(m)}_a &= \sum_{k,\pm} \frac{k}{S} A^{(m)\pm} Q^{m\pm}_a \end{aligned}$$

$$(D_a X)^{(m)} = \sum_{k,\pm} \frac{k}{S} (\delta X)^{(m)\pm} Q^{m\pm}_a \tag{A12}$$

where the k dependence of the harmonic coefficients is suppressed. We also often suppress m and $\pm$ indices for clarity. The other multipoles are expanded in analogy with I_{A_l}. The heat flux for each fluid component is given by $q_i = (\rho_i + p_i)v_i$, where v_i is the velocity, and the total heat flux is given by $q = \sum_i q_i$. We write the baryon velocity simply as v, and define a constant $B^{(m)}_0 \equiv \Pi^{(m)}_B/\rho_\gamma$ to quantify the magnetic field anisotropic stress source. In the frame in which $\Omega_a = 0$ gradients are purely scalar $(\delta \bar{X})^{(1)} = 0$.

3. Harmonic multipole equations

Expanded into harmonics, the photon multipole equations (A1) become

$$\begin{aligned} I'_l + \frac{k}{2l+1}\left[\frac{(l+1)^2-m^2}{l+1} I_{l+1} - l I_{l-1}\right] = \\ -Sn_e\sigma_T\left(I_l - \delta_{l0}I_0 - \frac{4}{3}\delta_{l1}v - \frac{2}{15}\zeta\delta_{l2}\right) \\ + \frac{8}{15}k\sigma\delta_{l2} - 4h'\delta_{l0} - \frac{4}{3}kA\delta_{l1} \end{aligned} \tag{A13}$$

where $l \geq m$, $I_0 = \delta\rho_\gamma/\rho_\gamma$, $I_l = 0$ for $l < m$, and m superscripts are implicit. The scalar source is $h' = (\delta S/S)'$. The equation for the neutrino multipoles (after neutrino decoupling) is the same but without the Thomson scattering terms (for massive neutrinos see Ref. [39]). The polarization multipole equations (A3) become

$$\begin{aligned} &\mathcal{E}^{m\pm\prime}_l + k\left[\frac{(l+3)(l-1)}{(l+1)^3}\frac{(l+1)^2-m^2}{(2l+1)}\mathcal{E}^{m\pm}_{l+1} - \frac{l}{2l+1}\mathcal{E}^{m\pm}_{l-1} - \frac{2m}{l(l+1)}\mathcal{B}^{m\mp}_l\right] = -Sn_e\sigma_T(\mathcal{E}^{m\pm}_l - \frac{2}{15}\zeta^{m\pm}\delta_{l2}) \\ &\mathcal{B}^{m\pm\prime}_l + k\left[\frac{(l+3)(l-1)}{(l+1)^3}\frac{(l+1)^2-m^2}{(2l+1)}\mathcal{B}^{m\pm}_{l+1} - \frac{l}{2l+1}\mathcal{B}^{m\pm}_{l-1} + \frac{2m}{l(l+1)}\mathcal{E}^{m\mp}_l\right] = 0. \end{aligned} \tag{A14}$$

4. Integral solutions

Solutions to the Boltzmann hierarchies can be found in terms of line of sight integrals. The I^m_l hierarchy has homogeneous solutions (i.e. solutions to Eq. A13 with RHS set to zero) given by derivatives of

$$\Psi^m_l(k\eta) \equiv \frac{l!}{(l-m)!}\frac{j_l(k\eta)}{(k\eta)^m} \tag{A15}$$

where $j_l(x)$ is a spherical Bessel function. These can be used to construct Green's function solutions to the full equations. For the polarization the result is less obvious, though solutions can easily be verified once found. For

vector modes ($m=1$) the solutions are [33]

$$I_l(\eta_0) \;=\; 4\int^{\eta_0} d\eta e^{-\tau}\left[Sn_e\sigma_T\bar{v}\Psi^1_l(\chi) + \left(k\bar{\sigma} + Sn_e\sigma_T\frac{\zeta}{4}\right)\frac{d\Psi^1_l(\chi)}{d\chi}\right] \tag{A16}$$

$$E^{\pm}_l(\eta_0) \;=\; \frac{l(l-1)}{l+1}\int^{\eta_0} d\eta Sn_e\sigma_T e^{-\tau}\left[\frac{1}{\chi}\frac{dj_l(\chi)}{d\chi} + \frac{j_l(\chi)}{\chi^2}\right]\zeta^{\pm} \tag{A17}$$

$$B^{\pm}_l(\eta_0) \;=\; -\frac{l(l-1)}{l+1}\int^{\eta_0} d\eta Sn_e\sigma_T e^{-\tau}\frac{j_l(\chi)}{\chi}\zeta^{\mp} \tag{A18}$$

where $\chi \equiv k(\eta_0-\eta)$. For tensors ($m=2$) the solutions are [27]

$$I_l(\eta_0) \;=\; 4\int^{\eta_0} d\eta e^{-\tau}\left[k\sigma + Sn_e\sigma_T\frac{\zeta}{4}\right]\Psi^2_l(\chi) \tag{A19}$$

$$E^{\pm}_l(\eta_0) \;=\; \frac{l(l-1)}{(l+1)(l+2)}\int^{\eta_0} d\eta Sn_e\sigma_T e^{-\tau}\left[\frac{d^2 j_l(\chi)}{d\chi^2} + \frac{4}{\chi}\frac{dj_l(\chi)}{d\chi} - \left(1-\frac{2}{\chi^2}\right)j_l(\chi)\right]\zeta^{\pm} \tag{A20}$$

$$B^{\pm}_l(\eta_0) \;=\; -2\frac{l(l-1)}{(l+1)(l+2)}\int^{\eta_0} d\eta Sn_e\sigma_T e^{-\tau}\left[\frac{dj_l(\chi)}{d\chi} + \frac{2}{\chi}j_l(\chi)\right]\zeta^{\mp}. \tag{A21}$$

Here τ is the optical depth from η to η_0, $\tau' = -Sn_e\sigma_T$.

5. Power spectra

Using the harmonic expansion of I_{A_l} in Eq. 18 the contribution to the C_l from type-m tensors becomes

$$C^{TT(m)}_l = \frac{\pi}{4}\frac{(2l)!}{(-2)^l(l!)^2}\sum_{k,k',\pm}\langle I^{\pm}_{l,k}I^{\pm}_{l,k'}\rangle Q^{\pm}_{A_l k}Q^{A_l\pm}_{k'}. \tag{A22}$$

The multipoles I_l can be related to some primordial variable $X_{A_m} = \sum_k\left(X^+Q^{m+}_{A_m} + X^-Q^{m-}_{A_m}\right)$ via a transfer function T^X_l defined by $I_l = T^X_l X$. Statistical isotropy and orthogonality of the harmonics implies that

$$\langle X^{\pm}_k X^{\pm}_{k'}\rangle = f_X(k)\delta_{kk'} \tag{A23}$$

where $\sum_k \delta_{kk'}Y_k = Y_{k'}$ and $f_X(k)$ is some function of the eigenvalue k. The normalization of the $Q^m_{A_l}$ is given by

$$\int dV Q^m_{A_l}Q^{mA_l} = \int dV\left(\frac{-S}{k}\right)^{l-m} Q^m_{A_m}D^{A_{l-m}}Q^m_{A_l} = \frac{(-2)^{l-m}(l+m)!(l-m)!}{(2l)!}N \tag{A24}$$

where we have integrated by parts repeatedly, then repeatedly applied the identity for the divergence (A11). The normalization is $N \equiv \int dV Q^{A_m}Q_{A_m}$. By statistical isotropy $C_l = (1/V)\int dV C_l$ and hence

$$C^{TT(m)}_l = \frac{\pi}{4}\frac{(l+m)!(l-m)!}{(-2)^m(l!)^2}\sum_{k,\pm}\frac{N}{V}|T^X_l(k)|^2 f_X(k) \tag{A25}$$

We choose to define a power spectrum $P_X(k)$ so that the real space isotropic variance is given by

$$\langle|X_{A_m}X^{A_m}|\rangle = \sum_{k,\pm}\frac{|N|}{V}f_X(k) \equiv \int d\ln k\; P_X(k) \tag{A26}$$

so the CMB power spectrum becomes

$$C^{TT(m)}_l = \frac{\pi}{4}\frac{(l+m)!(l-m)!}{2^m(l!)^2}\int d\ln k\; P_X(k)|T^X_l(k)|^2. \tag{A27}$$

Note that we have not had to choose a specific representation of Q_{A_m} or $\sum_k$.

The polarization C_l are obtained similarly [27] and in general we have

$$C_l^{JK(m)} = \frac{\pi}{4}\left[\frac{(l+1)(l+2)}{l(l-1)}\right]^{p/2}\frac{(2l)!}{(-2)^l(l!)^2}\frac{\langle J_{A_l}K^{A_l}\rangle}{\rho_\gamma^2} \tag{A28}$$

$$= \frac{\pi}{4}\left[\frac{(l+1)(l+2)}{l(l-1)}\right]^{p/2}\frac{(l+m)!(l-m)!}{2^m(l!)^2}\int d\ln k\; P_X(k)\, J_l^X K_l^X \tag{A29}$$

where JK is TT ($p=0$), EE or BB ($p=2$) or TE ($p=1$). Our conventions for the polarization are consistent with CMBFAST [5] and CAMB [29]. We have assumed a parity symmetric ensemble, so $C_l^{TB} = C_l^{EB} = 0$.

For tensors we use H_T where $h_{ij} = \sum_{k,\pm} 2H_T Q_{ij}^2$ and h_{ij} is the transverse traceless part of the metric tensor. This introduces an additional factor of 1/4 into the result for the C_l in terms of P_h and $T_l^{H_T}$.

The numerical factors in the hierarchy and C_l equations depend on the choice of normalization for the ℓ and k expansions. Neither $e_{\langle A_l\rangle}$ nor Q_{A_l} are normalized, so there are compensating numerical factors in the expression for the C_l. If desired one can do normalized expansions, corresponding to an ℓ-dependent re-scaling of the I_l and other harmonic coefficients, giving expressions in more manifest agreement with Ref. [32].

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